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On the Behavior of the Solution of the Burgers Equation as the Viscosity Goes to Zero

BUI AN TON

University of British Columbia, Vancouver, Canada

Submitted by J. L. Lions

The solution of the initial boundary-value problem

$$u_\epsilon' - \epsilon D^2 u_\epsilon + u_\epsilon D u_\epsilon = f \quad \text{on} \quad (a, b)\, x(0, T),$$
$$u_\epsilon(a, t) = u_\epsilon(b, t) = 0 \qquad \text{and} \qquad u_\epsilon(x, 0) = 0 \quad \text{on} \quad (a, b),$$

is shown to converge to the solution of the limiting equation as the viscosity tends to zero. Estimates on the rate of convergence are given.

The purpose of this paper is to study the behavior of the solution of the initial boundary-value problem:

$$\frac{\partial u_\epsilon}{\partial t} - \epsilon D^2 u_\epsilon + u_\epsilon D u_\epsilon = f \qquad \text{on} \qquad (a, b) \times (0, T), \quad u_\epsilon(a, t) = u_\epsilon(b, t) = 0,$$
$$u_\epsilon(x, 0) = 0,$$

as the viscosity ϵ tends to zero.

The above equation was introduced by Burgers [2–4], in the study of fluid dynamics. The behavior of the solution of the Cauchy's problem for the Burgers' equation as the viscosity goes to zero has been studied by Hopf [5].

The notations and the main result of the paper are given in Section 1. The proof of the theorem is carried out in Section 2.

1.

Let $G = (a, b)$ be a bounded open subset of R and let $D = \partial/\partial x$. We denote by $H = L^2(G)$ and by $(\cdot, \cdot)$ the inner product in H.

$W^{k,2}(G)$ is the real Hilbert space,

$$W^{k,2}(G) = \{u: u \text{ in } H, D^j u \text{ in } H, j \leqslant k\},$$

with the norm

$$\| u \|_{k,2} = \left\{ \sum_{0 \leqslant j \leqslant k} \| D^j u \|_H^2 \right\}^{1/2}$$

and the obvious inner product.

$W_0^{k,2}(G)$ is the completion of $C_0^\infty(G)$, the family of all infinitely differentiable functions with compact support in G, with respect to the $W^{k,2}(G)$-norm.

The dual of $W_0^{k,2}(G)$ is $W^{-k,2}(G)$.

$L^2(0, T; W^{k,2}(G))$ is the space of all equivalence classes of functions $u(t)$ from $(0, T)$ to $W^{k,2}(G)$ which are L^2-integrable over $(0, T)$. It is a real Hilbert space with the norm

$$\| u \|_{L^2(0,T;W^{k,2}(G))} = \left\{ \int_0^T \| u(t) \|_{k,2}^2 \, dt \right\}^{1/2}$$

and the usual inner product.

The main result of the paper is the following theorem.

THEOREM 1. *Let f be an element of $L^\infty(0, T; W_0^{3,2}(G))$. Then there exists a nonempty interval $(0, \tau)$ with τ independent of ϵ and $\tau < T$ such that:*

(i) $\| u_\epsilon(t) - u(t) \|_H = O(\epsilon^{1/2})$ *uniformly on* $(0, \tau)$.

(ii) $\| u_\epsilon - u \|_{L^\infty(0,\tau;W^{2,2}(G))} = O(\epsilon^{1/8})$.

u_ϵ is the unique solution of the initial boundary-value problem:

$$\begin{aligned} u_\epsilon' - \epsilon D^2 u_\epsilon + u_\epsilon D u_\epsilon &= f \qquad \text{on} \qquad G \times (0, T), \\ u_\epsilon(x, t) &= 0 \qquad \text{on} \qquad \partial G \times (0, T), \\ u_\epsilon(x, 0) &= 0 \qquad \text{on} \qquad G. \end{aligned}$$

u is the unique solution of:

$$u' + uDu = f \qquad \text{on} \qquad G \times (0, \tau),$$

$$u(x, t) = 0 \qquad \text{on} \qquad \partial G \times (0, \tau), \qquad u(x, 0) = 0.$$

2.

Let $\{w_j\}$ be a special basis for $W_0^{1,2}(G)$ with $-D^2 w_j = \lambda_j w_j$. It is clear that $\lambda_j > 0$ and $D^2 w_j$ is in $W_0^{1,2}(G)$. Thus w_j belongs to $W^{2,2}(G)$.

We have

$$D^4 w_j = \lambda_j^2 w_j \qquad \text{and} \qquad -D^6 w_j = \lambda_j^3 w_j .$$

So D^4w_j, D^2w_j are in $W_0^{1,2}(G)$ and w_j is in $W^{6,2}(G) \cap W_0^{1,2}(G)$. Set:

$$u_k(t) = \sum_{j=1}^{k} c_{jk}(t)\, w_j .$$

Consider the system of nonlinear ordinary differential equations in $c_{jk}(t)$:

$$(u'_{\epsilon k}, w_j) - \epsilon(D^2u_{\epsilon k}, w_j) + (u_{\epsilon k}Du_{\epsilon k}, w_j) = (f, w_j) \quad c_{jk}(0) = 0, \quad j = 1,..., k.$$

PROPOSITION 1. *Let f be as in Theorem 1. Then there exists a local solution $u_{\epsilon k}$ of the above system of ordinary differential equations.*

Proof. The existence of a local solution of the above system of ordinary differential equations follows from the Caratheodory's theorem.

LEMMA 1. *Let f be as in Theorem 1. Then there exists a nonempty interval $(0, \tau)$ with τ independent of both ϵ and k such that:*

$$\| u_{\epsilon k} \|_{L^\infty(0,\tau; W_0^{1,2}(G))} + \| u_{\epsilon k} \|_{L^\infty(0,\tau; W^{3,2}(G))} + \epsilon \| u_{\epsilon k} \|^2_{L^2(0,\tau; W^{4,2}(G))} \leqslant M.$$

M is independent of both ϵ and k.

Proof of Lemma 1. From Proposition 1, we know that there exists a local solution $u_{\epsilon k}$ in $L^2(0, T_k^\epsilon; W_0^{1,2}(G))$ of:

$$(u'_{\epsilon k}, w_j) - \epsilon(D^2u_{\epsilon k}, w_j) + (u_{\epsilon k}Du_{\epsilon k}, w_j) = (f, w_j), \qquad c_{jk}(0) = 0.$$

Moreover $u'_{\epsilon k}$, $D^2u_{\epsilon k}$, $D^4u_{\epsilon k}$ and $D^6u_{\epsilon k}$ are in $W_0^{1,2}(G)$.

We have by an easy argument:

$$-(u'_{\epsilon k}, D^6u_{\epsilon k}) + \epsilon(D^2u_{\epsilon k}, D^6u_{\epsilon k}) - (u_{\epsilon k}Du_{\epsilon k}, D^6u_{\epsilon k}) = -(f, D^6u_{\epsilon k}).$$

(1) Consider the expression $-(u'_{\epsilon k}, D^6u_{\epsilon k})$.

Integration by parts gives

$$-(u'_{\epsilon k}, D^6u_{\epsilon k}) = (Du'_{\epsilon k}, D^5u_{\epsilon k}) = -(D^2u'_{\epsilon k}, D^4u_{\epsilon k}) = (D^3u'_{\epsilon k}, D^3u_{\epsilon k}).$$

So:

$$-(u'_{\epsilon k}, D^6u_{\epsilon k}) = \frac{1}{2}\frac{d}{dt} \| D^3u_{\epsilon k}(t)\|_H^2 .$$

(2) Using the above remarks on $D^2u_{\epsilon k}$ and $D^4u_{\epsilon k}$, we get

$$(D^2u_{\epsilon k}, D^6u_{\epsilon k}) = -(D^3u_{\epsilon k}, D^5u_{\epsilon k}) = (D^4u_{\epsilon k}, D^4u_{\epsilon k}).$$

(3) An integration by parts gives

$$(u_{\epsilon k}Du_{\epsilon k}, D^6u_{\epsilon k}) = -(D(u_{\epsilon k}Du_{\epsilon k}), D^5u_{\epsilon k}).$$
$$= -(Du_{\epsilon k}Du_{\epsilon k} + u_{\epsilon k}D^2u_{\epsilon k}, D^5u_{\epsilon k}).$$

Since $D^4u_{\epsilon k}$ is in $W_0^{1,2}(G)$, an integration by parts yields:

$$\begin{aligned}(u_{\epsilon k}Du_{\epsilon k}, D^6u_{\epsilon k}) &= (2Du_{\epsilon k} \cdot D^2u_{\epsilon k} + D^2u_{\epsilon k} \cdot Du_{\epsilon k} + u_{\epsilon k}D^3u_{\epsilon k}, D^4u_{\epsilon k}).\\ &= 3(Du_{\epsilon k} \cdot D^2u_{\epsilon k}, D^4u_{\epsilon k}) + (u_{\epsilon k}D^3u_{\epsilon k}, D^4u_{\epsilon k}).\\ &= E_1 + E_2.\end{aligned}$$

Consider E_1. Since $D^2u_{\epsilon k}$ is in $W_0^{1,2}(G)$ and $u_{\epsilon k}$ is in $W^{6,2}(G)$, it is easy to see that $Du_{\epsilon k}D^2u_{\epsilon k}$ is in $W_0^{1,2}(G)$.

An integration by parts gives

$$E_1 = 3(Du_{\epsilon k}D^2u_{\epsilon k}, D^4u_{\epsilon k}) = -3(D^2u_{\epsilon k}D^2u_{\epsilon k}, D^3u_{\epsilon k}) - 3(Du_{\epsilon k}D^3u_{\epsilon k}, D^3u_{\epsilon k}).$$

The Sobolev imbedding theorem gives

$$W^{3,2}(G) \subset W^{2,4}(G) \qquad \text{and} \qquad W^{2,2}(G) \subset C(clG).$$

Therefore,

$$| E_1 | \leqslant C \| u_{\epsilon k} \|^3_{W^{3,2}(G)}.$$

C is independent of both ϵ and k.

Finally consider

$$E_2 = (u_{\epsilon k}D^3u_{\epsilon k}, D^4u_{\epsilon k}).$$

Again we may integrate by parts. We obtain

$$E_2 = -(Du_{\epsilon k}D^3u_{\epsilon k} + u_{\epsilon k}D^4u_{\epsilon k}, D^3u_{\epsilon k}).$$

An elementary computation gives

$$(u_{\epsilon k}D^4u_{\epsilon k}, D^3u_{\epsilon k}) = -\tfrac{1}{2}(D^3u_{\epsilon k}, Du_{\epsilon k}D^3u_{\epsilon k}).$$

Thus,

$$E_2 = -(Du_{\epsilon k}D^3u_{\epsilon k}, D^3u_{\epsilon k}) + \tfrac{1}{2}(D^3u_{\epsilon k}, Du_{\epsilon k}D^3u_{\epsilon k}).$$
$$| E_2 | \leqslant C \| u_{\epsilon k}(t) \|^3_{W^{3,2}(G)}.$$

C is again independent of both ϵ and k.

Therefore,

$$\frac{1}{2}\frac{d}{dt} \| D^3u_{\epsilon k}(t) \|^2_H \leqslant C\{\| u_{\epsilon k}(t) \|^3_{W^{3,2}(G)} + \| f(t) \|_{W_0^{3,2}(G)} \| u_{\epsilon k}(t) \|_{W^{3,2}(G)}\}.$$

The different constants C are all independent of both ϵ and k.

(4) Since $u_{\epsilon k}$ is in $W_0^{1,2}(G) \cap W^{3,2}(G)$, we have

$$\| u_{\epsilon k} \|_{W^{3,2}(G)} \leqslant M\{\| u_{\epsilon k} \|_H + \| D^3 u_{\epsilon k} \|_H\}.$$

But $u_{\epsilon k}$ and $D^2 u_{\epsilon k}$ are in $W_0^{1,2}(G)$. Thus the Poincaré inequality gives

$$\| u_{\epsilon k} \|_H \leqslant C \| D u_{\epsilon k} \|_H \qquad \text{and} \qquad \| D^2 u_{\epsilon k} \|_H \leqslant C_2 \| D^3 u_{\epsilon k} \|_H .$$

Moreover we have by an easy argument,

$$\| D u_{\epsilon k} \|_H \leqslant C_3 \| D^2 u_{\epsilon k} \|_H .$$

Thus,

$$\| u_{\epsilon k} \|_{W^{3,2}(G)} \leqslant M \| D^3 u_{\epsilon k} \|_H .$$

Therefore,

$$\frac{d}{dt} \| D^3 u_{\epsilon k}(t) \|_H \leqslant C\{\| f(t) \|_{W^{3,2}(G)} + \| D^3 u_{\epsilon k}(t) \|_H^2\}.$$

We also have

$$D^3 u_{\epsilon k}(x, 0) = 0.$$

(5) Let $\varphi(t)$ be the solution of the initial-value problem

$$\frac{d\varphi}{dt} = C\{\varphi^2 + \| f(t) \|_{W^{3,2}(G)}\}, \qquad \varphi(0) = 0.$$

It is well known that the above initial-value problem has a local solution φ, continuous in $[0, \tau]$ with $\tau < T$. It is clear that ϵ is independent of both ϵ and k.

A standard theorem of ordinary differential equations gives:

$$\| D^3 u_{\epsilon k}(t) \|_H \leqslant \varphi(t) \qquad \text{for } t \text{ in } \min\{(0, \tau), (0, T_k^{\epsilon})\}.$$

By continuation, we have

$$\| D^3 u_{\epsilon k}(t) \|_H \leqslant \varphi(t) \qquad \text{for t in } (0, \tau).$$

It follows from the above arguments that

$$\| u_{\epsilon k}(t) \|_{W^{3,2}(G)} \leqslant C \| D^3 u_{\epsilon k}(t) \|_H \leqslant M.$$

Thus,

$$\| u_{\epsilon k} \|_{L^\infty(0,\tau; W^{3,2}(G))} \leqslant M.$$

M is a constant independent of both ϵ and k.

All the other assertions of the lemma are now trivial to show.

THEOREM 2. *Let f be as in Theorem 1. Then there exists a unique solution u_ϵ of the initial boundary-value problem:*

$$u_\epsilon' - \epsilon D^2 u_\epsilon + u_\epsilon D u_\epsilon = f \qquad on \qquad G \times (0, T),$$
$$u_\epsilon(x, 0) = 0, \qquad u_\epsilon(a, t) = u_\epsilon(b, t) = 0.$$

Moreover there exists a nonempty interval $(0, \tau)$ independent of ϵ such that:

$$\| u_\epsilon \|_{L^\infty(0,\tau; W^{3,2}(G))} + \| u_\epsilon \|_{L^\infty(0,\tau; W_0^{1,2}(G))} + \epsilon \| u_\epsilon \|^2_{L^2(0,\tau; W^{4,2}(G))}$$
$$+ \| u_\epsilon' \|_{L^2(0,\tau; W^{-1,2}(G))} \leqslant C.$$

C is independent of ϵ.

Proof of Theorem 2. (1) Let P_k be the orthogonal projection of H onto $[w_1, ..., w_k]$, i.e.,

$$P_k h = \sum_{j=1}^{k} (h, w_j)\, w_j .$$

P_k is a bounded linear operator mapping $W_0^{1,2}(G)$ into $W_0^{1,2}(G)$ and since $P_k^* = P_k$, P_k is a bounded linear mapping of $W^{-1,2}(G)$ into $W^{-1,2}(G)$.

Let $u_{\epsilon k}$ be as in Lemma 1. We have

$$u_{\epsilon k}' - \epsilon P_k(D^2 u_{\epsilon k}) + P_k(g_{\epsilon k}) = P_k f$$

where $((g_{\epsilon k}, \varphi)) = (u_{\epsilon k} D u_{\epsilon k}, \varphi)$ with φ in $W_0^{1,2}(G)$. $((\cdot, \cdot))$ is the pairing between $W_0^{1,2}(G)$ and its dual.

It is now easy to see that

$$\| u_{\epsilon k}' \|_{L^2(0,\tau; W^{-1,2}(G))} \leqslant C.$$

C is a constant independent of both ϵ and k.

(2) From the weak compactness of the unit ball in a reflexive Banach space we get by taking subsequences if necessary: $u_{\epsilon k} \to u_\epsilon$ in the weak*-topology of $L^\infty(0, \tau; W^{3,2}(G))$, $u_{\epsilon k} \to u_\epsilon$ in the weak*-topology of $L^\infty(0, \tau; W_0^{1,2}(G))$, $\epsilon^{1/2} u_{\epsilon k} \to \epsilon^{1/2} u_\epsilon$ weakly in $L^2(0, \tau; W^{4,2}(G))$ and $u_{\epsilon k}' \to u_\epsilon'$ weakly in $L^2(0, \tau; W^{-1,2}(G))$ as $k \to \infty$.

It follows from Aubin's theorem that $u_{\epsilon k} \to u_\epsilon$ in $L^2(0, \tau; W^{3,2}(G))$. Thus applying the Lebesgue convergence theorem we get

$$u_{\epsilon k} D u_{\epsilon k} \to u_\epsilon D u_\epsilon \qquad \text{in} \qquad L^2(0, \tau; H).$$

Hence,

$$-\int_0^\tau (u_\epsilon, \varphi' w_j)\,dt - \int_0^\tau \epsilon(D^2 u_\epsilon, \varphi w_j)\,dt + \int_0^\tau (u_\epsilon D u_\epsilon, \varphi w_j)\,dt = \int_0^\tau (f, \varphi w_j)\,dt$$

for all φ in $C^1(0, \tau)$ with $\varphi(\tau) = 0$.

Now it is standard to show that:

$$u_\epsilon' - \epsilon D^2 u_\epsilon + u_\epsilon D u_\epsilon = f \qquad \text{on} \qquad G \times (0, \tau),$$
$$u_\epsilon(x, 0) = 0 \qquad \text{and} \qquad u_\epsilon(a, t) = u_\epsilon(b, t) = 0.$$

It is easy to show that the solution is unique.

The estimates of the theorem follow from those of Lemma 1.

Proof of Theorem 1. (1) Let u_ϵ be the unique solution of the initial boundary-value problem of Theorem 2. From the weak compactness of the unit ball in a reflexive Banach space, we obtain by taking subsequences if necessary: $u_\epsilon \to u$ in the weak*-topology of $L^\infty(0, \tau; W_0^{1,2}(G))$, $u_\epsilon \to u$ in the weak*-topology of $L^\infty(0, \tau; W^{3,2}(G))$, $\epsilon^{1/2} u_\epsilon \to 0$ weakly in $L^2(0, \tau; W^{4,2}(G))$ and $u_\epsilon' \to u'$ weakly in $L^2(0, \tau; W^{-1,2}(G))$ as $\epsilon \to 0$.

Aubin's theorem gives $u_\epsilon \to u$ in $L^2(0, \tau; W^{2,2}(G))$ as $\epsilon \to 0$. Applying the Lebesgue convergence theorem, we get

$$u_\epsilon D u_\epsilon \to u D u \qquad \text{in} \qquad L^2(0, \tau; H).$$

Thus,

$$u' + u D u = f, \qquad u(x, 0) = 0 \qquad \text{and} \qquad u(a, t) = u(b, t) = 0.$$

(2) We have

$$\frac{1}{2}\frac{d}{dt}\| u_\epsilon(t) - u(t)\|_H^2 + \epsilon(D u_\epsilon, D(u_\epsilon - u)) + (u_\epsilon D u_\epsilon - u D u, u_\epsilon - u) = 0.$$

So

$$\frac{1}{2}\frac{d}{dt}\| u_\epsilon(t) - u(t)\|_H^2 + \frac{1}{2}\epsilon \| D(u_\epsilon - u)\|_H^2 \leqslant \tfrac{1}{2}\epsilon \| D u \|_H^2 + |((u_\epsilon - u)\, D u_\epsilon, u_\epsilon - u)| + |(u D(u_\epsilon - u), u_\epsilon - u)| \,.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\| u_\epsilon(t) - u(t)\|_H^2 + \frac{1}{2}\epsilon \| D(u_\epsilon - u)\|_H^2 \leqslant \tfrac{1}{2}\epsilon \| D u_\epsilon \|_H^2 + \| u_\epsilon - u \|_H^2 \| D u_\epsilon \|_{L^\infty(G)} + |(u D(u_\epsilon - u), u_\epsilon - u)| \,.$$

An integration by parts yields

$$(uD(u_\epsilon - u), u_\epsilon - u) = -\tfrac{1}{2}((u_\epsilon - u), (u_\epsilon - u)\, Du).$$

Hence:

$$\frac{d}{dt} \| u_\epsilon(t) - u(t)\|_H^2 \leqslant \epsilon \| Du \|_H^2 + C \| u_\epsilon - u \|_H^2,$$
$$\leqslant M\epsilon + C \| u_\epsilon - u \|_H^2 .$$

We have also,

$$\| u_\epsilon(0) - u(0)\|_H = 0.$$

Therefore,

$$\| u_\epsilon(t) - u(t)\|_H^2 \leqslant M\epsilon\{\exp(t/2C) - 1\}/2C.$$

On the other hand, it is known that

$$\| v \|_{W^{2,2}(G)} \leqslant \eta \| v \|_{W^{3,2}(G)} + \eta^{-2} \| v \|_H .$$

Take $\eta = \epsilon^{1/8}$, we obtain

$$\| u_\epsilon - u \|_{L^2(0,T;W^{2,2}(G))} = O(\epsilon^{1/8}).$$

The theorem is proved.

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